

Fully answer all the questions in the booklets provided. Closed book. No calculators. No communication of any kind with your fellow students. The only aid allowed is a single $8\frac{1}{2} \times 11$ hand written sheet. Time: 2 hours.

Question 1. Determine the radii of convergence of each of the following power series.

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$$\begin{array}{ll} \text{(a)} \sum_{n=1}^{\infty} x^n \sin\left(\frac{1}{2^n}\right) & \text{(b)} \sum_{n=1}^{\infty} n!(x-5)^n \\ \text{(c)} \sum_{n=1}^{\infty} \frac{x^n}{(n!)^3} & \text{(d)} \sum_{n=1}^{\infty} \frac{x^n}{3^n \sqrt{n}} \\ \text{(e)} \sum_{n=1}^{\infty} \frac{x^{2n}}{4^n n^2} & \end{array}$$

Question 2. Determine the interval of convergence of each of the following power series.

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$$\begin{array}{lll} \text{(a)} \sum_{n=1}^{\infty} \sqrt{n} x^n & \text{(b)} \sum_{n=1}^{\infty} \frac{x^{2n}}{n^2} & \text{(c)} \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^n (n+3)} \end{array}$$

Question 3. Suppose $f(x)$ is a function which has Taylor series

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$$f(x) = 2x - \frac{1}{2}x^2 + 5x^3 + \dots$$

Calculate, up to and including terms of order x^3 , the Taylor expansions of the functions

$$\begin{array}{lll} \text{(a)} f(x) \sin x & \text{(b)} f(x) + 2\sqrt{1-x} & \text{(c)} x f'(x). \end{array}$$

Question 4. Find the Taylor series, up to and including terms of order x^2 , of the function

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$$y = \ln(1 + 2 \sin x).$$

Question 5. Find the general solution to the following differential equations.

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$$\begin{array}{ll} \text{(a)} \frac{dy}{dx} = \frac{y(x^2 - 4)}{x} & \text{(b)} \frac{1}{t} \frac{dy}{dt} - y + 2 = 0. \end{array}$$

Question 6. Find the general solution to the following differential equations.

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$$\begin{array}{ll} \text{(a)} \frac{dy}{dx} + \frac{4x}{1+x^2} y = \frac{4x}{1+x^2} & \text{(b)} x \frac{dy}{dx} + (1+x)y - 1 = 0. \end{array}$$

Question 7. Solve the following initial value problems.

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$$\begin{array}{ll} \text{(a)} \frac{dy}{dx} + y = x^2, \quad y(1) = 2 & \text{(b)} \frac{dy}{dt} = \frac{y-1}{t-1}, \quad y(3) = 2. \end{array}$$