University of Saskatchewan

EE 480.3 Digital Control Systems

Final Examination, April 14, 2010

3 hour open-book exam
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1. (25) A continuous time system is defined as follows for input $x(t)$ and output $y(t)$.

$$Y(s) = G(s)X(s), \quad G(s) = \frac{1}{s(s+1)(s+2)}.$$

We apply unity gain negative feedback to this system, and adjust gain K. The discrete time model of this system with zero-order-hold ZOH for a sampling time $T_s = 0.01$ sec. is calculated as

$$G(z) = \frac{1.6542 \times 10^{-7}(z+0.2659)(z+3.7042)}{(z-1)(z-0.9900)(z-0.9802)}.$$

1. Show if the unity gain feedback applied to $G(s)$, i.e. $\dfrac{KG(s)}{1+KG(s)}$ is stable or not by a method to check stability.
2. The discrete time transfer function $G(z)$ was mapped to the w-domain by the bilinear transformation as given below. Apply Ruth-Hurwitz's stability criterion to find the stable range of the gain K.

$$G(w) = \frac{-0.005w + 1}{w^3 + 3w^2 + 2w}$$

2. (25) A second order digital control system, $G(s) = \dfrac{1}{s(s+1)}$ is shown in Fig. P2 (a). The discrete model of the open loop system combined with a ZOH (Zero Order Hold) has been calculated for $T = 0.1$ as follows:

$$G(z) = 0.0048\frac{z+0.9672}{(z-1)(z-0.9048)}.$$

The root locus was plotted for the following compensated system with a phase lead compensator having a pole at $z = 0$ and a zero at $z = 0.7$.

$$G_c(z) = K\frac{(z+0.9672)(z-0.7)}{z(z-1)(z-0.9048)}.$$

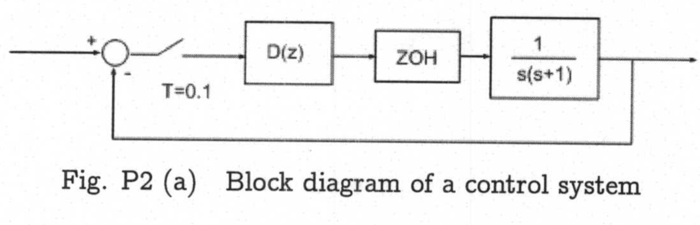


Fig. P2 (a) Block diagram of a control system

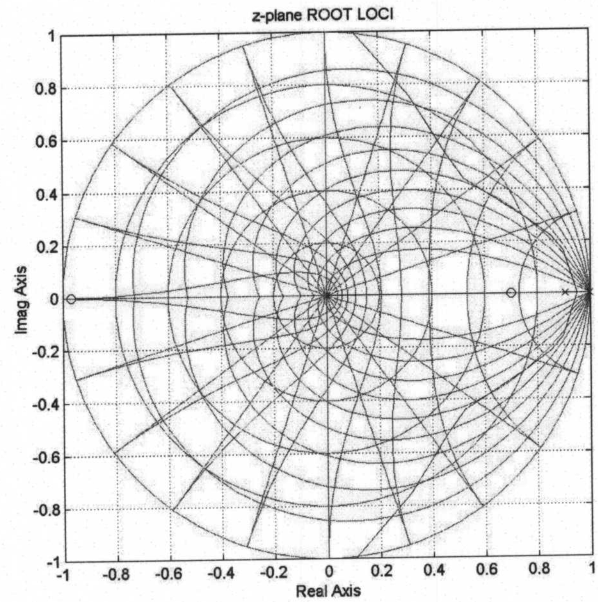


Fig. P2 (b) Root Locus plot of the phase lead compensated system

At the gain $K = 0.323$, the closed loop poles found from the root locus are $z = 0.4608$ and $z = 0.5605 \pm j0.4005$. Calculate the following values with respect to the dominant poles of the closed loop system.

1. Damping ratio ζ
2. % Overshoot
3. Time constant τ
4. Natural frequency ω_n
5. Damped natural frequency ω_d
6. Settling time for 3% criterion T_s

3. (25) An analog system with the transfer function,

$$G(s) = \frac{1}{s^2 + s + 1}$$

has state feedback to form a feedback control system as shown in Fig. P3. The state variables are x_1 and x_2. The sampling rate is 10 samples per second.

1. Using the state variables defined by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, obtain the analog state equation of $G(s)$ which takes the form of

$$\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$$

and the output equation which gives $y(t)$.

2. Obtain the state transition matrix $\Phi(t)$ from **A**.
3. Obtain the discrete time state equation expressed in the form of

$$\mathbf{x}(k+1) = \mathbf{P}\mathbf{x}(k) + \mathbf{q}u(k)$$

To calculate P and **q**, consider only up to the second order term of the Taylor series expansion,

$$\mathbf{P} = \Phi(T) = \mathbf{I} + \mathbf{A}T + \mathbf{A}^2\frac{T^2}{2!} + \mathbf{A}^3\frac{T^2}{3!} + \cdots$$

4. Determine the state feedback gain vector **K**= $[K_1, K_2]^T$ that makes the system in Figure 2 critical damping with the two poles at $z = 0.9$ by using the Ackermann's formula.

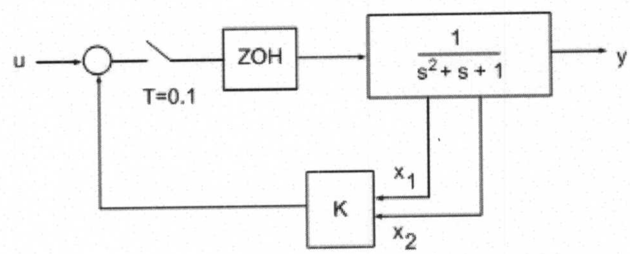


Fig. P3 A state feedback control system

4. (25) An antenna positioning system shown if Fig. P4 (a) consists of a physical system,

$$G(s) = \frac{100}{s(s+6)}$$

and a first order digital compensator given by

$$D(z) = K_d\frac{z - z_0}{z - z_p} \iff D(w) = \frac{a_1 w + a_0}{b_1 w + 1}$$

The digital compensator $D(z)$ is transformed by the bilinear transformation into the w-domain as $D(w)$ to facilitate design.

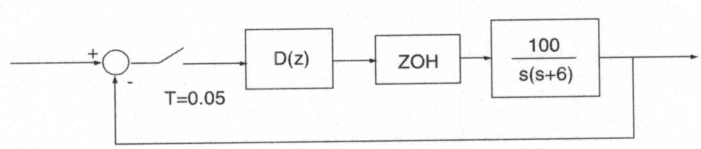


Fig. P4 (a) Block diagram of an antenna control system

The antenna positioning system preceded with a ZOH (Zero Order Hold) is described by the step invariant z-transform as

$$G(z) = 0.1134\frac{z + 0.9049}{(z-1)(z-0.7408)}$$

The frequency response of $G(z)$ is plotted as Bode diagram for the frequency ω_w as shown in Fig. P4 (b).

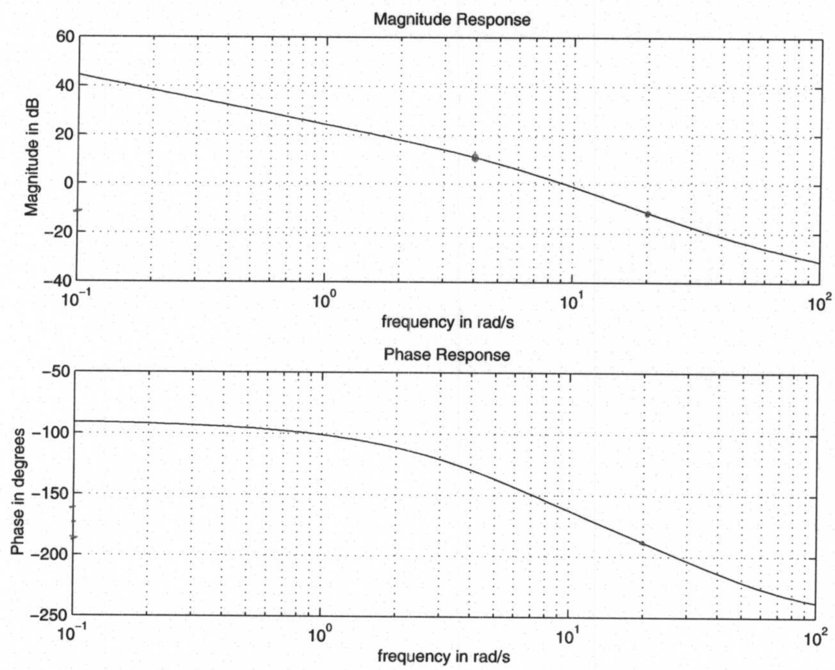


Fig. P4 (b) Bode diagram with respect to the ω_w frequency axis.

1. Design a phase-lead compensator which satisfies the phase margin $\phi_m = 45°$. Assume $a_0 = 1$. Test if all three criteria are met if a 0 dB cross-over frequency of $\omega_{w_1} = 20$ rad/sec is chosen. Calculate the compensator transfer function $D(w)$ in the w-domain only (Do not attempt to calculate $D(z)$)
2. Design a phase-lag compensator which satisfies the phase margin $\phi_m = 45°$ at 0 dB cross-over frequency of $\omega_{w_1} = 4$ rad/sec. Assume $a_0 = 1$. The phase-lag compensator specific method (simpler method) is acceptable. Write the compensator transfer function $D(w)$ only.

5. (25) The note on the state observer - the excerpt from EE480 notes is given in Appendix. Referring to the Appendix, answer the following questions

1. What is the purpose of using the observer in the state feedback systems?
2. Designing the observer means to determine the matrix $\mathbf{G}$. The Ackermann's formula is utilized to determine $\mathbf{G}$. Explain how to choose the characteristic equation of the error dynamics,

$$\alpha_e(z) = |z\mathbf{I} - (\mathbf{A} - \mathbf{GC})| = |z\mathbf{I} - \mathbf{A} + \mathbf{GC})| = 0.$$

3. The block diagram of the observer is given at the end of the notes. Using this block diagram, complete the block diagram of the state feedback control system that incorporates the observer.

Appendix: The State Observer

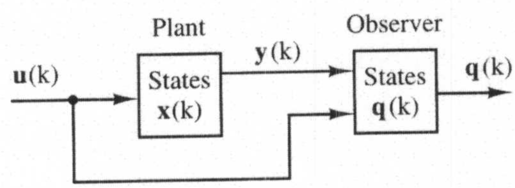
Observer Model

Our plant is described by

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{Ax}(k) + \mathbf{Bu}(k) \\ \mathbf{y}(k) &= \mathbf{Cx}(k) \end{aligned}$$

Using the z-transform

$$\begin{aligned} z\mathbf{X}(z) &= \mathbf{AX}(z) + \mathbf{BU}(z) \\ \mathbf{X}(z) &= (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}(z) \end{aligned}$$



The observer is a system to observe (estimate) the state $\mathbf{x}(k+1)$ from input $\mathbf{u}(k)$ and output $\mathbf{y}(k)$. Thus, the equation of the observer is

$$\begin{aligned} \mathbf{q}(k+1) &= \mathbf{Fq}(k) + \mathbf{Gy}(k) + \mathbf{Hu}(k) \\ \mathbf{Q}(z) &= (z\mathbf{I} - \mathbf{F})^{-1}[\mathbf{GY}(z) + \mathbf{HU}(z)] \end{aligned}$$

From

$$\mathbf{Y}(z) = \mathbf{CX}(z), \quad \mathbf{X}(z) = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}(z)$$

$$\begin{aligned} \mathbf{Q}(z) &= (z\mathbf{I} - \mathbf{F})^{-1}[\mathbf{GY}(z) + \mathbf{HU}(z)] \\ \mathbf{Q}(z) &= (z\mathbf{I} - \mathbf{F})^{-1}[\mathbf{GCX}(z) + \mathbf{HU}(z)] \\ \mathbf{Q}(z) &= (z\mathbf{I} - \mathbf{F})^{-1}[\mathbf{GC}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{H}]\mathbf{U}(z) \end{aligned}$$

The state estimation is to make $\mathbf{Q}(z) = \mathbf{X}(z)$ for the same input $\mathbf{U}(z)$.

$$\mathbf{Q}(z) = \mathbf{X}(z) = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}(z)$$

Thus,

$$(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = (z\mathbf{I} - \mathbf{F})^{-1}[\mathbf{GC}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{H}]$$

$$\begin{aligned} (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} &= (z\mathbf{I} - \mathbf{F})^{-1}[\mathbf{GC}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{H}] \\ (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} &= (z\mathbf{I} - \mathbf{F})^{-1}\mathbf{GC}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + (z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H} \end{aligned}$$

$$[\mathbf{I} - (z\mathbf{I} - \mathbf{F})^{-1}\mathbf{GC}](z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = (z\mathbf{I} - \mathbf{F})^{-1}\mathbf{H}$$

$$(z\mathbf{I}-\mathbf{F})^{-1}[z\mathbf{I}-(\mathbf{F}+\mathbf{GC})](z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}=(z\mathbf{I}-\mathbf{F})^{-1}\mathbf{H}$$

$$[z\mathbf{I}-(\mathbf{F}+\mathbf{GC})](z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}=\mathbf{H}$$

$$(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}=[z\mathbf{I}-(\mathbf{F}+\mathbf{GC})]^{-1}\mathbf{H}$$

If we choose the input matrix of the plant $\mathbf{B}$ equal to that of the estimator, $\mathbf{H}$, then

$$\mathbf{A}=\mathbf{F}+\mathbf{GC}$$

The observer state equation is rewritten as

$$\begin{aligned}\mathbf{q}(k+1) &= \mathbf{Fq}(k)+\mathbf{Gy}(k)+\mathbf{Hu}(k)\\ \mathbf{q}(k+1) &= (\mathbf{A}-\mathbf{GC})\mathbf{q}(k)+\mathbf{Gy}(k)+\mathbf{Bu}(k)\end{aligned}$$

This is the equation of a prediction observer. $\mathbf{q}(k+1)$ is the predicted estimate of $\mathbf{x}(k)$. The prediction observer must satisfy the characteristic equation,

$$|z\mathbf{I}-\mathbf{A}+\mathbf{GC}|=0$$

Errors in Estimation

The error in the state estimation process is

$$\mathbf{e}(k)=\mathbf{x}(k)-\mathbf{q}(k)$$

$$\begin{aligned}\mathbf{e}(k+1) &= \mathbf{x}(k+1)-\mathbf{q}(k+1)\\ &= \mathbf{Ax}(k)+\mathbf{Bu}(k)-[(\mathbf{A}-\mathbf{GC})\mathbf{q}(k)+\mathbf{Gy}(k)+\mathbf{Hu}(k)]\\ &= \mathbf{Ax}(k)+\mathbf{Bu}(k)-[(\mathbf{A}-\mathbf{GC})\mathbf{q}(k)+\mathbf{GCx}(k)+\mathbf{Bu}(k)]\\ &= (\mathbf{A}-\mathbf{GC})[\mathbf{x}(k)-\mathbf{q}(k)]\\ &= (\mathbf{A}-\mathbf{GC})\mathbf{e}(k)\end{aligned}$$

Thus, the error dynamics have the same characteristic equation as that of the observer,

$$|z\mathbf{I}-(\mathbf{A}-\mathbf{GC})|=0$$

All the matrices in the observer state equation except $\mathbf{G}$ are from the plant state equation. $\mathbf{G}$ can be independently determined from the characteristic equation of the error dynamics,

$$\alpha_e(z)=|z\mathbf{I}-(\mathbf{A}-\mathbf{GC})|=|z\mathbf{I}-\mathbf{A}+\mathbf{GC})|=0$$

Observer Design

Similar to the design of a state feed back system, the observer requires to determine the unknown matrix $\mathbf{G}$ from $\alpha_e(z)$.

$$\det\,[z\mathbf{I}-\mathbf{A}+\mathbf{GC}]=0,$$

$$\det\,[z\mathbf{I}-\mathbf{A}+\mathbf{GC}]=\alpha_e(z).$$