EE.456: Digital Communications
MIDTERM EXAMINATION, 5:00PM–7:00PM, October 25, 2004
(2 hours, closed book)

Examiner: Ha H. Nguyen

Permitted Materials: **Calculator**

Note: **There are 3 questions. All questions are of equal value (with part marks indicated) but not necessarily of equal difficulty. Full marks shall only be given to solutions that are properly explained and justified.**

1\. The noise $X(t)$ applied to a linear filter in Figure 1 is modeled as a wide-sense stationary (WSS) random process with power spectral density $G_X(f)$. Let $Y(t)$ denote the noise process at the output of the filter.

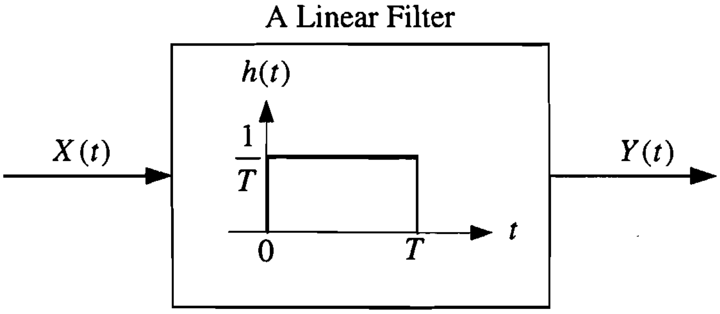


Figure 1: System under consideration in Question 1.

[1] (a) Is $Y(t)$ wide-sense stationary noise process? Why?

[2] (b) Show that the frequency response of the filter in Figure 1 is:

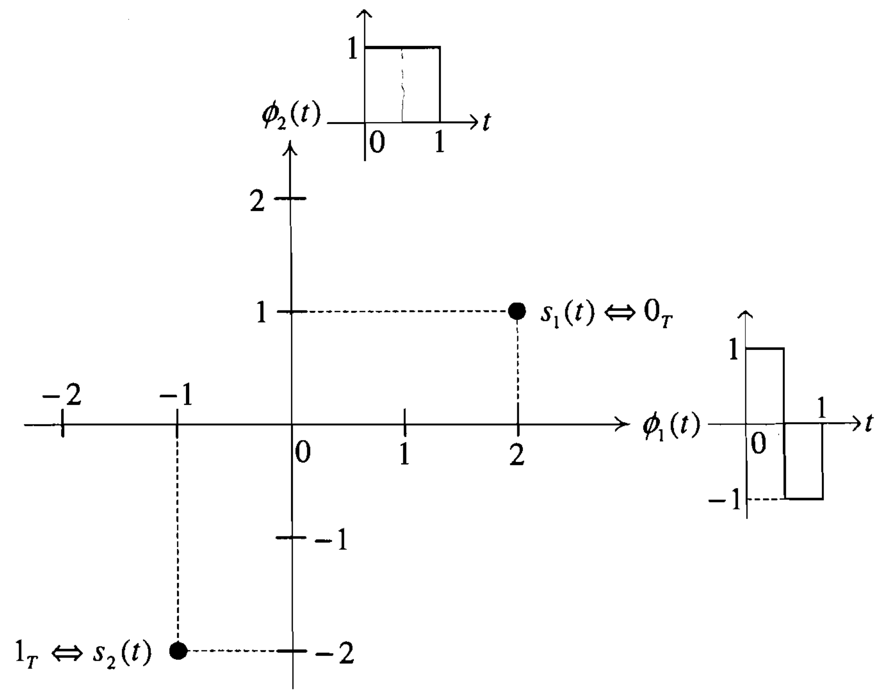
$$H(f) = \frac{\sin(\pi f T)}{\pi f T} e^{-j\pi f T}$$

[1] (c) If m_X is the dc component in $X(t)$, what is the dc component in $Y(t)$?

[4] (d) If $X(t)$ is a zero-mean, *white* noise process with power spectral density $N_0/2$, find the power spectral density of the noise process $Y(t)$. What frequency components cannot be present in the output process? Explain your answer.

[2] (e) Suppose that the output noise is sampled every T_s seconds to obtain the noise samples $Y(kT_s)$, $k = 0, 1, 2, \ldots$. Find the smallest values of T_s so that the noise samples are *uncorrelated*. Explain your answer.

2. Consider the signal space diagram shown in the Figure below.



[2] (a) Determine and sketch the two signals $s_1(t)$ and $s_2(t)$.

[3] (b) The two signals $s_1(t)$ and $s_2(t)$ are used for the transmission of equally likely bits 0 and 1, respectively, over an additive white Gaussian noise (AWGN) channel. Clearly draw the decision boundary and the decision regions of the optimum receiver. Write the expression for the optimum decision rule.

[2] (c) Find and sketch the two orthonormal basis functions $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ such that the optimum receiver can be implemented using only the projection $\hat{r}_2$ of the received signal $r(t)$ onto the basis function $\hat{\phi}_2(t)$. Draw the block diagram of such a receiver that uses a matched filter.

[3] (d) Consider now the following argument put forth by your classmate. She reasons that since the component of the signals along $\hat{\phi}_1(t)$ is not useful at the receiver in determining which bit was transmitted, one should not even transmit this component of the signal. Thus she modifies the transmitted signal as follows:

$$\begin{aligned} s_1^M(t) &= s_1(t) - \Big(\text{component of } s_1(t) \text{ along } \hat{\phi}_1(t)\Big) \\ s_2^M(t) &= s_2(t) - \Big(\text{component of } s_2(t) \text{ along } \hat{\phi}_1(t)\Big) \end{aligned}$$

Clearly identify the locations of $s_1^M(t)$ and $s_2^M(t)$ in the signal space diagram. What is the average energy of this signal set? Compare it to the average energy of the original set. Comment.

3. In antipodal signalling, two signals $s(t)$ and $-s(t)$ are used to transmit equally likely bits 0 and 1, respectively. Consider two communication systems, called system-(i) and system-(ii), that use two different time-limited waveforms as shown in Figure 2.

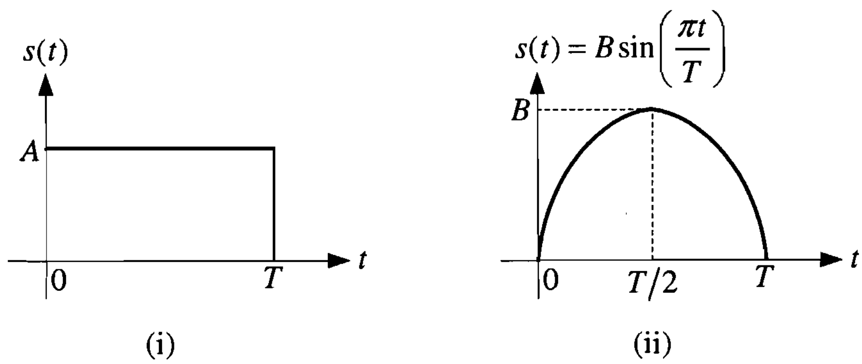


Figure 2: Two waveforms for $s(t)$.

[4] (a) What is the relationship between the parameters A and B of the two waveforms if the two communication systems have the same error performance? Explain your answer.

[3] (b) Since the two signals in Figure 2 are time-limited, they cannot be band-limited and a bandwidth definition is required. Here we define W to be the bandwidth of the signal $s(t)$ if 95% of the total energy of $s(t)$ is contained inside the band $[-W, W]$. The bandwidths of the above two signals have been found in Assignment 3 to be $W_1 = 2.07/T$ and $W_2 = 0.91/T$. Assume that both systems have the same bit rate of 2 Mbps. What are the required bandwidths of the two systems? What system is preferred and why?

[3] (c) Consider the system that uses $s(t)$ in Figure 2-(i). How large does the voltage level A need to be set to achieve an error probability of 10^{-6} if the bit rate is 2 Mbps and $\frac{N_0}{2} = 10^{-8}$ (watts/Hz)?

POTENTIALLY USEFUL FACTS:

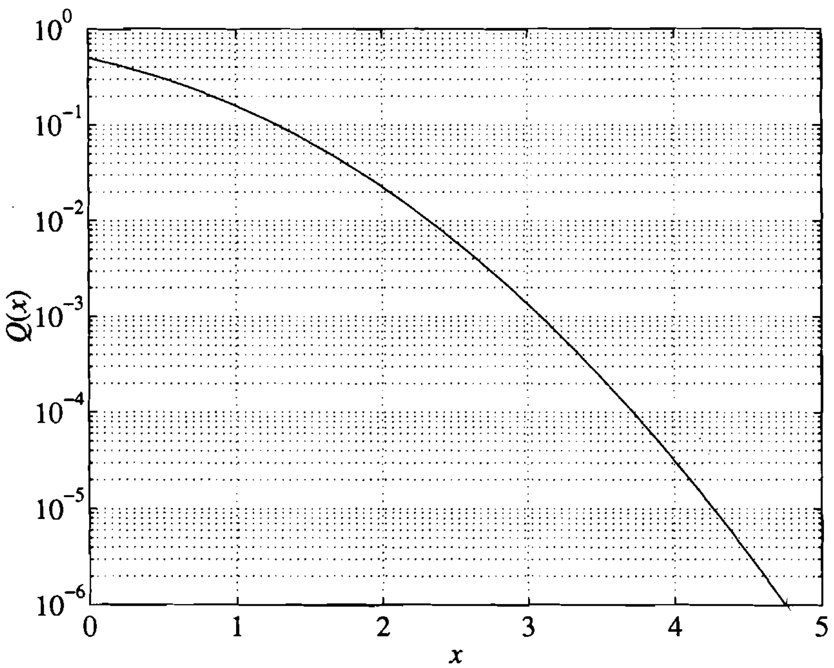
- Fourier transform and its properties:

$$x(t) \overset{\mathcal{FT}}{\longleftrightarrow} X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-j2\pi ft}\mathrm{d}t \tag{1}$$

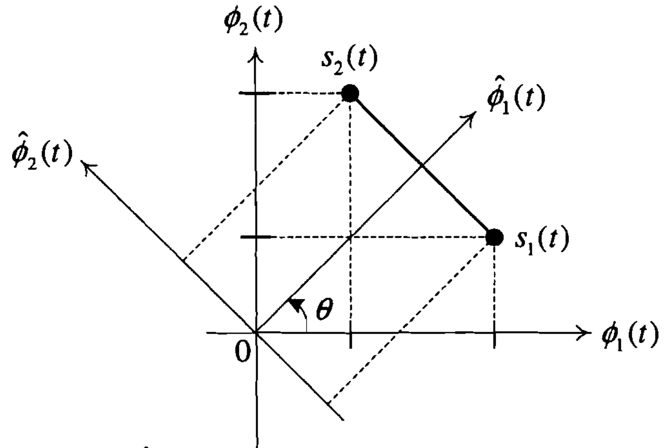
$$x(t-T) \overset{\mathcal{FT}}{\longleftrightarrow} X(f)\mathrm{e}^{-j2\pi fT} \tag{2}$$

$$\begin{cases} 1-\dfrac{|t|}{T} & \text{for } |t| \leq T \\ 0 & \text{for } |t| > T \end{cases} \overset{\mathcal{FT}}{\longleftrightarrow} T\left[\frac{\sin(\pi fT)}{\pi fT}\right]^2 \tag{3}$$

- Plot of the Q-function:



- Rotation of axes:



$$\begin{pmatrix} \hat{\phi}_1(t) \\ \hat{\phi}_2(t) \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix} \tag{4}$$

SOLUTIONS TO MIDTERM EXAMINATION (Fall 2004)

1. The noise $X(t)$ applied to a linear filter in Figure 1 is modeled as a wide-sense stationary (WSS) random process with power spectral density $G_X(f)$. Let $Y(t)$ denote the noise process at the output of the filter.

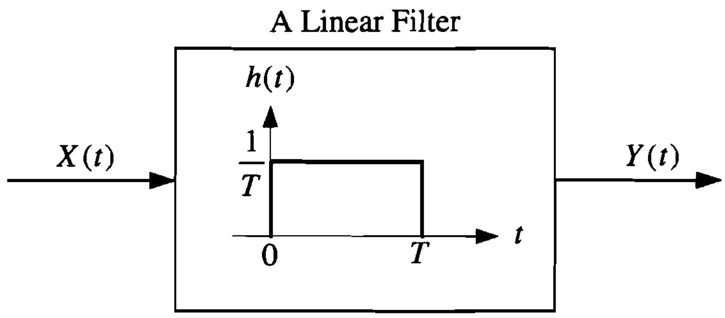


Figure 1: System under consideration in Question 1.

[1] (a) The system is LTI. The input noise process $X(t)$ is WSS. The output noise process $Y(t)$ therefore is also WSS.

[2] (b) The frequency response of the filter in Figure 1 is:

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t)\mathrm{e}^{-j2\pi ft}\mathrm{d}t = \int_0^T \frac{1}{T}\mathrm{e}^{-j2\pi ft}\mathrm{d}t \\ &= \frac{1}{T}\frac{1}{-j2\pi f}\,\mathrm{e}^{-j2\pi ft}\Big|_0^T = -\frac{1}{j2\pi fT}\left[\mathrm{e}^{-j2\pi fT}-1\right] \\ &= -\frac{\mathrm{e}^{-j\pi fT}-\mathrm{e}^{j\pi fT}}{j2\pi fT}\mathrm{e}^{-j\pi fT} \\ &= \frac{j2\sin(\pi fT)}{j2\pi fT}\mathrm{e}^{-j\pi fT} = \frac{\sin(\pi fT)}{\pi fT}\mathrm{e}^{-j\pi fT} = \mathrm{sinc}(fT)\mathrm{e}^{-j\pi fT} \end{aligned}$$

[1] (c) If m_X is the dc component in $X(t)$, then the dc component in $Y(t)$ is

$$m_Y = m_X \times H(0) = m_X \times \mathrm{sinc}(0) = m_X$$

[4] (d) If $X(t)$ is a zero-mean, *white* noise process with power spectral density $N_0/2$, the power spectral density of $Y(t)$ is given by:

$$G_Y(f) = G_X(f)|H(f)|^2 = \frac{N_0}{2}\left[\frac{\sin(\pi fT)}{\pi fT}\right]^2 = \frac{N_0}{2}\mathrm{sinc}^2(fT)$$

The frequency components that are not present in the output process are the components that make $G_Y(f) = 0$. Obviously, these components correspond to $\pi fT = k\pi$, or $f = k/T$, $k = \pm 1, \pm 2, \ldots$.

[2] (e) Suppose that the output noise is sampled every T_s seconds to obtain the noise samples $Y(kT_s)$, $k = 0, 1, 2, \ldots$. Find the smallest value of T_s so that the noise samples are *uncorrelated.*

To answer this question, one needs to find the autocorrelation function of the output process $Y(t)$, which is given as the inverse Fourier transform of $G_Y(f)$. Using the Fourier transform pair provided, one has:

$$R_Y(\tau) = \mathcal{F}^{-1}\{G_Y(f)\} = \begin{cases} \frac{N_0}{2T}\left(1 - \frac{|t|}{T}\right) & \text{for } |t| \le T \\ 0 & \text{for } |t| > T \end{cases}$$

The autocorrelation function of $Y(t)$ is plotted in Figure 2. The smallest value of the sampling period T_s to yield *uncorrelated* noise samples is the smallest positive value of τ that makes $R_Y(\tau) = 0$. From Figure 2 the answer is obviously $T_s = T$.

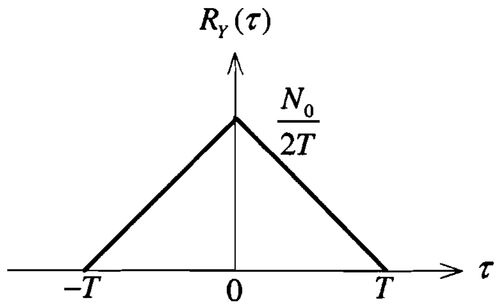


Figure 2: Plot of $R_Y(\tau)$.

[2] 2. (a) The two signals $s_1(t)$ and $s_2(t)$ are simply determined from their coordinates as:

$$\begin{aligned} s_1(t) &= s_{11}\phi_1(t) + s_{12}\phi_2(t) = 2\phi_1(t) + \phi_2(t) \\ s_2(t) &= s_{21}\phi_1(t) + s_{22}\phi_2(t) = -\phi_1(t) - 2\phi_2(t) \end{aligned}$$

The two signals are plotted in Figure 3.

[3] (b) Since the two binary bits are equally likely, the decision boundary of the optimum receiver is the bisector of the line joining the two signals. Such optimum decision boundary and the decision regions are shown in Figure 4. A simple inspection of the decision boundary in Figure 4 gives the following optimum decision rule:

$$r_1 \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} -r_2$$

Of course, the above expression for the optimum decision rule can be also be reached by substituting all the signal coordinates into the following "fundamental" minimum distance rule:

$$(r_1 - s_{21})^2 + (r_2 - s_{22})^2 \underset{\text{“}1_D\text{”}}{\overset{\text{“}0_D\text{”}}{\gtreqless}} (r_1 - s_{11})^2 + (r_2 - s_{12})^2$$

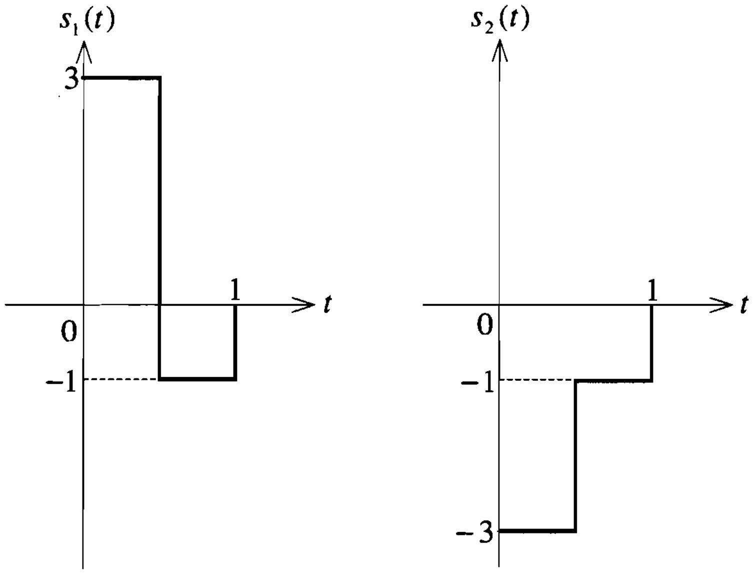


Figure 3: Plots of $s_1(t)$ and $s_2(t)$.

[2] (c) From the signal space diagram in Figure 4, it is clear that the orthonormal basis functions $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ are obtained by rotating $\phi_1(t)$ and $\phi_2(t)$ by 45^0 clockwise (i.e., $\theta = -\frac{\pi}{4}$), or but 175^0 clockwise (i.e., $\theta = \frac{3\pi}{4}$). This rotation is to ensure that $\hat{\phi}_1(t)$ is perpendicular to the line joining the two signals. Choosing $\theta = -\frac{\pi}{4}$ yields:

$$\begin{bmatrix} \hat{\phi}_1(t) \\ \hat{\phi}_2(t) \end{bmatrix} = \begin{bmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}$$

It follows that

$$\hat{\phi}_1(t) = \frac{1}{\sqrt{2}}[\phi_1(t) - \phi_2(t)]$$
$$\hat{\phi}_2(t) = \frac{1}{\sqrt{2}}[\phi_1(t) + \phi_2(t)]$$

These two functions are plotted in Figure 5 together with the block diagram of a receiver that uses one matched filter.

[3] (d) Consider now the following argument put forth by your classmate. She reasons that since the component of the signals along $\hat{\phi}_1(t)$ is not useful at the receiver in determining which bit was transmitted, one should not even transmit this component of the signal. Thus she modifies the transmitted signal as follows:

$$s_1^M(t) = s_1(t) - \Big(\text{component of } s_1(t) \text{ along } \hat{\phi}_1(t)\Big)$$
$$s_2^M(t) = s_2(t) - \Big(\text{component of } s_2(t) \text{ along } \hat{\phi}_1(t)\Big)$$

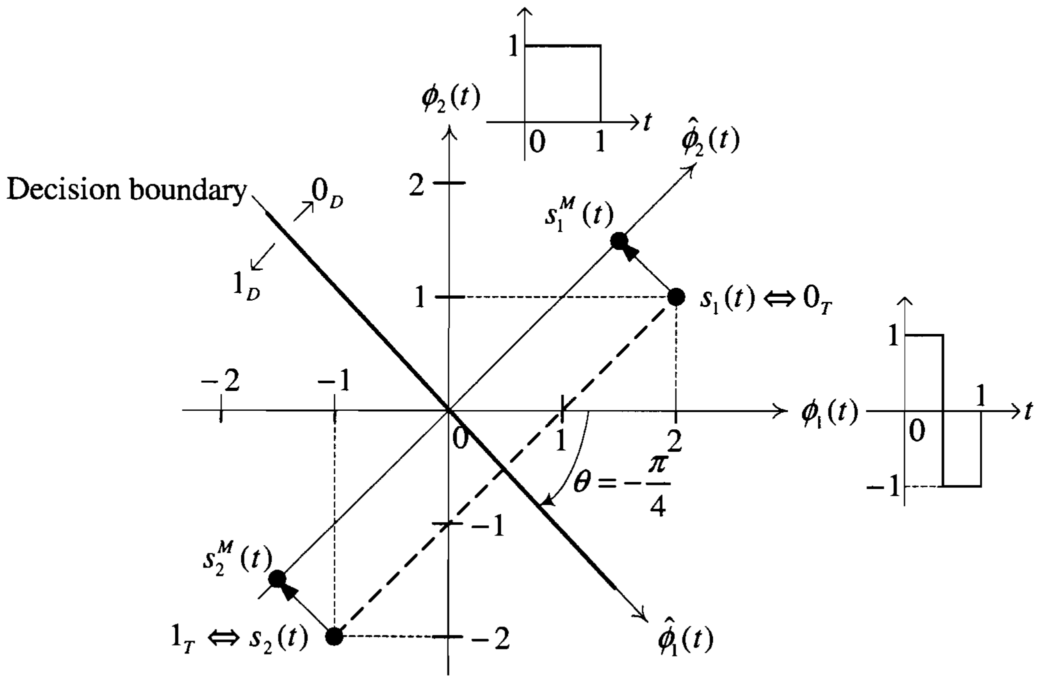


Figure 4: Optimum decision boundary and decision regions.

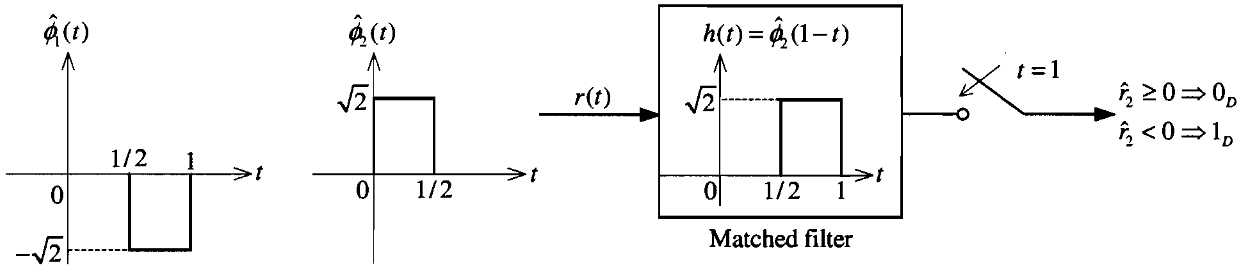


Figure 5: Plots of $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ and the simplified receiver.

The locations of $s_1^M(t)$ and $s_2^M(t)$ are clearly shown on the signal space diagram in Figure 4. The average energy of this signal set is simply:

$$\begin{aligned} E^M &= \frac{1}{2}(\hat{s}_{12}^2 + \hat{s}_{22}^2) = \frac{1}{2}[(E_1 - \hat{s}_{11}^2) + (E_2 - \hat{s}_{21}^2)] \\ &= \frac{1}{2}(E_1 + E_2) - \frac{1}{2}(\hat{s}_{11}^2 + \hat{s}_{21}^2) = 5 - \left(\frac{1}{\sqrt{2}}\right)^2 = 4.5 \text{ (joules)} \end{aligned}$$

The average energy of the modified signal set is clearly smaller that the the average energy of the original set (which is $\frac{E_1+E_2}{2} = 5$ joules). Since the distance between the modified signals is the same at that of the original signals, both sets perform indentically in terms of the bit error rate (BER) performance. The modified set is therefore preferred due to its better energy (or power) efficiency.

3. In antipodal signalling, two signals $s(t)$ and $-s(t)$ are used to transmit equally likely bits 0 and 1, respectively. Consider two communication systems, called system-(i) and system-(ii), that use two different time-limited waveforms as shown in Figure 6.

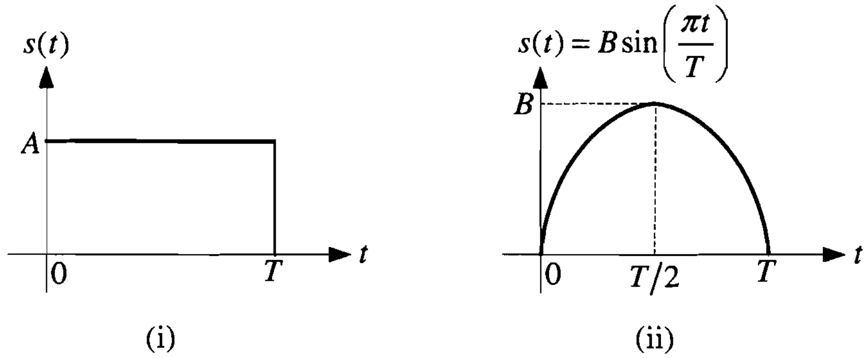


Figure 6: Two waveforms for $s(t)$.

[4] (a) When the two bits are equally likely (uniform source), the error performance of a digital binary communications system only depends on the Euclidean distance between the two signals. This distance is $2\sqrt{E}$ for antipodal signalling, where $\sqrt{E}$ is the energy of $s(t)$.

It follows that, for the two communication systems to have the same error performance, the energies of the two waveforms in Figure 6 have to be the same. That is:

$$A^2T = \frac{B^2T}{2} \Rightarrow \boxed{B = A\sqrt{2}}$$

[3] (b) Define W to be the bandwidth of the signal $s(t)$ if 95% of the total energy of $s(t)$ is contained inside the band $[-W, W]$. The bandwidths of the above two signals have been found in Assignment 3 to be $W_1 = 2.07/T$ and $W_2 = 0.91/T$.

Since both systems have the same bit rate of 2 Mbps, the bit duration in both system is $T = 1/(2 \times 10^6) = 0.5 \times 10^{-6}$ seconds.

The required bandwidths of the two systems are therefore $W_1 = 2.07/(0.5 \times 10^{-6}) = 4.14 \times 10^6$ Hz $= 4.14$ MHz and $W_2 = 0.91/(0.5 \times 10^{-6}) = 1.82 \times 10^6$ Hz $=$ 1.82 MHz.

Clearly, when the two systems have the same error performance, system-(ii) is preferred since it requires less transmission bandwidth.

[3] (c) Consider the system that uses $s(t)$ in Figure 6-(i). Then the error probability is

$$\Pr[\text{error}] = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) = 10^{-6}$$

$$\Rightarrow \quad \frac{2A^2T}{N_0} = [Q^{-1}(10^{-6})]^2 = 4.75^2$$

$$\Rightarrow \quad A = \left(\frac{4.75^2 N_0}{2T}\right)^{1/2} = \left(\frac{4.75^2 \times 2 \times 10^{-8}}{2 \times 0.5 \times 10^{-6}}\right)^{1/2} = 0.6718 \text{ (volts)}$$