EE301 Midterm Solutions 2002

1.a) From the 1st Maxwell equation

$$\rho = \epsilon_0 \vec{\nabla}\cdot\vec{E} = \frac{\epsilon_0}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right)$$

$$= \frac{Q\epsilon_0}{r^2}\frac{\partial}{\partial r}\left(1-\left(\frac{r^2}{2}+r+1\right)e^{-r}\right)$$

$$= \frac{Q\epsilon_0}{r^2}\left[\left(\frac{r^2}{2}+r+1\right)e^{-r}-(r+1)e^{-r}\right]$$

$$\Rightarrow \quad \rho = \frac{Q\epsilon_0}{2}e^{-r}$$

b) at $r=1$, $\theta=\pi/4$, $\varphi=\pi/4$ $\qquad \vec{E} = Q\left(1-\frac{5}{2}e^{-1}\right)\vec{a}_r$

let $\quad A = \left(1-\frac{5}{2}e^{-1}\right)Q = 8.03\times10^{-2}Q$

to convert to Cartesian coordinates, dot with the unit vectors

$$E_x = \vec{E}\cdot\vec{a}_x = A\,\vec{a}_r\cdot\vec{a}_x = A\sin\theta\cos\varphi = \frac{1}{2}A$$

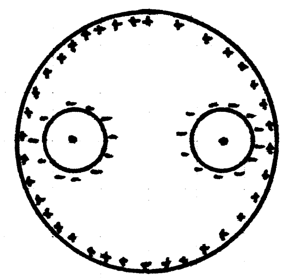
$$E_y = \vec{E}\cdot\vec{a}_y = A\,\vec{a}_r\cdot\vec{a}_y = A\sin\theta\sin\varphi = \frac{1}{2}A$$

$$E_z = \vec{E}\cdot\vec{a}_z = A\,\vec{a}_r\cdot\vec{a}_z = A\cos\theta = \frac{1}{\sqrt{2}}A$$

so

$$\vec{E} = \left(4.02\times10^{-2}Q\,\vec{a}_x + 4.02\times10^{-2}Q\,\vec{a}_y + 5.68\times10^{-2}Q\,\vec{a}_z\right)$$

2. a) Recall that for any metal subject to static electric fields, $\vec{E} = 0$ inside the metal. The positive charge Q inside each cavity draws negative charge to the cavity surface to terminate the field lines from the charge. In order to keep the metal overall neutral, there must be 2Q of positive charge distributed over the sphere's surface. Inside the cavities, Gauss's Law and the spherical symmetry of the cavities ensures that the the field will be that of the point charge



$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\vec{a}_r \qquad r < 5\,\text{cm}$$

taking the origin of the spherical coordinate system as the position of the charge.

b) There is no force between the charges since the electric field is screened by the metal. There could be a force between Q and the surface charge on the cavity surface but by symmetry that force must also be zero. $\therefore$ there is no force on the charges.

c) The field outside the sphere is due to the positive charge on the surface. What goes on inside the cavities does not matter outside except that the charges Q are the origin of the surface charge. The surface charge is distributed uniformly (because of symmetry) so we can use Gauss's Law to determine the field.

$$\vec{E} = E_r \vec{a}_r \quad \text{by symmetry} \quad E_r \text{ indep. of } \theta, \varphi$$

so over a spherical shell of radius $r > 20\text{cm}$

$$\epsilon_0 \oint \vec{E} \cdot d\vec{S} = 4\pi r^2 \epsilon_0 E_r = 2Q$$

So $$\vec{E} = \frac{Q}{2\pi\epsilon_0 r^2} \vec{a}_r \qquad r > 20\text{cm}$$

3. a) The energy stored in the capacitor is

$$W = \frac{1}{2} CV^2 = \frac{1}{2}(10^{-9} F)(10 v.)^2 = 5 \times 10^{-8} J$$

b) The charge on the 1nF cap. is originally

$$Q = CV = (10^{-9} F)(10 v.) = 10^{-8} C$$

After the 2nF cap. is connected, the charge redistributes between the two caps. The voltage across each cap. is V' and

for the 1nF cap. $Q_1 = C_1 V'$

for the 2nF cap. $Q_2 = C_2 V'$

The total charge doesn't change so

$$10^{-8} C = Q_1 + Q_2 = (C_1 + C_2) V'$$

or

$$V' = \frac{10^{-8} C}{C_1 + C_2} = \frac{10^{-8} C}{3nF} = 3.33 V.$$

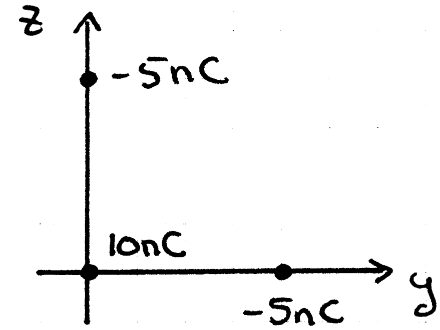
c) The total energy is now

$$W = \frac{1}{2} C_1 V'^2 + \frac{1}{2} C_2 V'^2 = \frac{1}{2}(10^{-9} F + 2 \times 10^{-9} F)(3.33 V)^2$$

$$= 1.67 \times 10^{-8} J$$

d) When the charge redistributes, currents must flow in the wires connecting the capacitors. Joule heating due to the finite resistance of the wires accounts for most of the missing energy. There can also be losses in the dielectric of the capacitors. If the wires have no resistance (superconductors) and there is no dielectric then the charge will flow back & forth between the caps, oscillating forever.

4. Calculate the electric field vector at (1,1,1) for each charge separately and use superposition to find the total field vector



$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$\vec{r} = \vec{a}_x + \vec{a}_y + \vec{a}_z$

for the charge at (0,0,0) $\vec{r}' = 0$ $\quad \vec{r}-\vec{r}' = \vec{a}_x + \vec{a}_y + \vec{a}_z$

$$\vec{E}_1 = \frac{10nC}{4\pi\epsilon_0} \frac{\vec{a}_x + \vec{a}_y + \vec{a}_z}{3^{3/2}}$$

for the charge at (0,0,1) $\vec{r}' = \vec{a}_z$ $\quad \vec{r}-\vec{r}' = \vec{a}_x + \vec{a}_y$

$$\vec{E}_2 = \frac{-5nC}{4\pi\epsilon_0} \frac{\vec{a}_x + \vec{a}_y}{2^{3/2}}$$

for the charge at (0,1,0) $\vec{r}' = \vec{a}_y$ $\quad \vec{r}-\vec{r}' = \vec{a}_x + \vec{a}_z$

$$\vec{E}_3 = \frac{-5nC}{4\pi\epsilon_0} \frac{\vec{a}_x + \vec{a}_z}{2^{3/2}}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + E = \frac{5nC}{4\pi\epsilon_0}\left[\left(\frac{2}{3^{3/2}} - \frac{2}{2^{3/2}}\right)\vec{a}_x + \left(\frac{2}{3^{3/2}} - \frac{1}{2^{3/2}}\right)\vec{a}_y + \left(\frac{2}{3^{3/2}} - \frac{1}{2^{3/2}}\right)\vec{a}_z\right]$$

$$= -14.5\,\vec{a}_x + 1.41\,\vec{a}_y + 1.41\,\vec{a}_z \quad V/m$$

5. Due to the spherical symmetry all fields will be purely radial and will depend only on r. The displacement field is determined by the free charge

$$\oint \vec{D}\cdot d\vec{s} = Q_{free,encl.} = \int_V \rho \, dV$$

$$4\pi r^2 D_r = 4\pi \int_0^r A(1 - r'/a) r'^2 dr'$$

$$D_r = \frac{A}{r^2}\left(\frac{r'^3}{3} - \frac{r'^4}{4a}\right)\Big|_0^r = A\left(\frac{r}{3} - \frac{r^2}{4a}\right)$$

so

$$\vec{D} = Ar\left(\frac{1}{3} - \frac{r}{4a}\right)\vec{a}_r$$

the electric field is simply $\vec{E} = \dfrac{\vec{D}}{\epsilon_0 \epsilon_R}$

so

$$\vec{E} = \frac{Ar}{\epsilon_0 \epsilon_R}\left(\frac{1}{3} - \frac{r}{4a}\right)\vec{a}_r$$

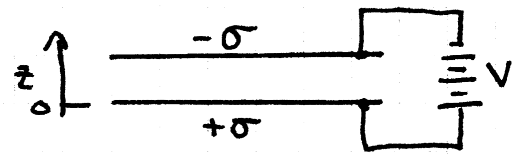
and the Polarization field is $\vec{P} = \vec{D} - \epsilon_0 \vec{E}$

so

$$\vec{P} = Ar\left(\frac{1}{3} - \frac{r}{4a}\right)\left(1 - \frac{1}{\epsilon_R}\right)\vec{a}_r$$

6. Let the charge density on the plates be σ. From Gauss's Law we know that (far from an edge)

$$\vec{D} = \sigma\, \vec{a}_z$$



and the electric field is

$$\vec{E} = \vec{D}/\epsilon_0 \epsilon_R = \frac{\sigma}{\epsilon_0 \epsilon_R}\, \vec{a}_r$$

ϵ_R is a function of z $\quad \epsilon_R = \epsilon_1 + (\epsilon_2 - \epsilon_1)\frac{z}{d}$

Determine the potential difference by integrating the electric field

$$V = -\int_d^0 E \cdot dl = -\int_d^0 \frac{\sigma}{\epsilon_0} \frac{1}{\epsilon_1 + (\epsilon_2 - \epsilon_1)\frac{z}{d}}\, dz$$

change varables in the integral to $x = \epsilon_1 + (\epsilon_2 - \epsilon_1)\frac{z}{d}$

$$dx = \frac{\epsilon_2 - \epsilon_1}{d}\, dz$$

so

$$V = -\frac{\sigma}{\epsilon_0} \frac{d}{(\epsilon_2 - \epsilon_1)} \int_{\epsilon_2}^{\epsilon_1} \frac{1}{x}\, dx$$

so

$$V = \frac{\sigma}{\epsilon_0} \frac{d}{(\epsilon_2 - \epsilon_1)} \ln \frac{\epsilon_2}{\epsilon_1}$$

the capacitance is

$$C = \frac{Q}{V} = \frac{A\sigma}{V} = \frac{A(\epsilon_2 - \epsilon_1)\epsilon_0}{d \ln \epsilon_2/\epsilon_1}$$